The Quantized Hall Insulator: a "Quantum" Signature of a "Classical" Transport Regime?

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Experimental studies of the transitions from a primary quantum Hall (QH) liquid at filling factor $\nu = 1/k$ (with k an odd integer) to the insulator have indicated a "quantized Hall insulator" (QHI) behavior: while the longitudinal resistivity diverges with decreasing temperature and current bias, the Hall resistivity remains quantized at the value kh/e^2. We review the experimental results and the theoretical studies addressing this phenomenon. In particular, we discuss a theoretical approach which employs a model of the insulator as a random network of weakly coupled puddles of QH liquid at fixed ν. This model is proved to exhibit a robust quantization of the Hall resistivity, provided the electron transport on the network is *incoherent*. Subsequent theoretical studies have focused on the controversy whether the assumption of incoherence is necessary. The emergent conclusion is that in the quantum coherent transport regime, quantum interference destroys the QHI as a consequence of localization. Once the localization length becomes much shorter than the dephasing length, the Hall resistivity diverges. We conclude by mentioning some recent experimental observations and open questions.

I. INTRODUCTION

A. The concept of "Hall Insulator"

An electric insulator is defined as a state of the electronic system where charge transport is strongly suppressed. In an ideal insulator, an infinite voltage is required in order to pass current through the system. However, physical insulators are studied experimentally in conditions where the temperature (T), the frequency of the driving source (ω) and the voltage bias (V) are finite, and typically do not exhibit an ideal insulating behavior. It is nevertheless possible to classify a quantum many–body electronic state as an "insulator", provided its resistance for current flowing along the direction of an applied voltage diverges in the limit where $T, \omega, V \to 0$ and the system size is infinite.

In the presence of a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, the conductivity tensor acquires an off-diagonal component σ_{xy}. An insulating behavior is then characterized by the vanishing of σ_{xx} and σ_{xy} and correspondingly a divergence of the longitudinal resistivity ρ_{xx}, which distinguishes it quite unambiguously from a conducting state. In contrast, the *Hall resistance*

$$\rho_{xy} = \frac{\sigma_{yx}}{(\sigma_{xx}^2 + \sigma_{xy}^2)} \tag{1}$$

is a subtle quantity: its behavior in the limit $T, \omega, V \to 0$ depends crucially on the ratio σ_{xy}/σ_{xx} when both components approach zero. In particular, in the case where the scaling relation $\sigma_{xy} \sim \sigma_{xx}^2$ holds, ρ_{xy} is *finite*. Such a peculiar insulating state, in which the Hall resistance is essentially indistinguishable from a conductor, is named a "Hall Insulator".

A number of theoretical studies[1,2,3,4,5] have argued that a Hall insulating behavior, where $\rho_{xy} \sim B$ as in a Drude conductor, is actually a quite generic property of disordered non–interacting electron systems. These correspond to electronic states where the insulating character is a consequence of Anderson localization; the behavior of ρ_{xy} possibly marks their distinction from band insulators or Mott (interaction–dominated) insulators. However, the dominant role of disorder in this case poses a difficulty: a theoretical evaluation of the transport coefficients involves a disorder averaging procedure, and is hence quite subtle. Indeed, the existence of a Hall Insulator was challenged by Entin–Wohlman *et al.*,[6] who showed that a direct derivation of the resistivity tensor (as opposed to inversion of a calculated conductivity tensor) yields an exponentially divergent ρ_{xy} for $T \to 0$. They further argue that this behavior is to be expected in a "true d.c." measurement. Experimental studies fail to settle this controversy: while some data[7] support the claims of Ref. [6], other experiments have reported an observation of a Hall Insulator.[8]

B. The Insulating Phase in the Quantum Hall Regime

The behavior of the Hall resistance is of particular significance in the so called "quantum Hall (QH) regime", which characterizes the low–T behavior of disordered two–dimensional (2D) electron systems subject to a strong perpendicular magnetic field. These systems exhibit a rich phase diagram indicating a multitude of transitions between phases with distinct transport properties.[9] These include primarily the various QH phases, characterized by quantized values of the Hall resistivity: $\rho_{xy} = h/e^2\nu$ in a wide range of carrier densities n and magnetic fields B centered around certain rational values of the filling factor $\nu = n\phi_0/B$ (where $\phi_0 = hc/e$ is the flux quantum). These plateaus in ρ_{xy} are accompanied by a *vanishing* longitudinal resistivity ρ_{xx}. At sufficiently strong magnetic field or disorder, the series of QH–to–QH transitions is terminated by a transition to an insulator, marked by a *divergence* of ρ_{xx}.

Based on the flux–attachment mapping of electrons in the QH regime to "composite (Chern–Simons) bosons", and a resulting set of laws of corresponding states, Kivelson, Lee and Zhang (KLZ)[10] have argued that the above mentioned insulating phase should exhibit the Hall Insulator behavior $\rho_{xy} \sim B/nec$. Their theory, however, does not provide a clean derivation of this expression for ρ_{xy} except at the particular quantized values of ν. The classical, Drude–like linear dependence on B is therefore presented as a suggestive interpolation. This behavior is consistent with the experimen-

tal data of Goldman *et al.*,[11] in apparent contradiction with the exponential divergence of ρ_{xy} reported by Willett *et al.*[12] at low ν. The latter, however, can possibly be interpreted as evidence for a Wigner crystal, which is expected to form at sufficiently low filling factors. These studies indicate that the insulating state in a strong magnetic field is not necessarily unique: the Hall resistance may serve as a probe which effectively distinguishes between different mechanisms for an insulating behavior.

II. EXPERIMENTAL EVIDENCE FOR A QUANTIZED HALL INSULATOR

Significant insight on the nature of the insulating phase in the QH regime was gained by later experimental studies of the transition from the primary QH states $\nu = 1/k$ (where k is an odd integer) to the neighboring insulator, driven by an increasing magnetic field B. In particular, a striking resemblance of the transport properties in the vicinity of the transition to the behavior near a superconductor–to–insulator transition in thin films[13] have inspired a further investigation of these transitions in view of the theoretical framework of bosonic Chern-Simons used by KLZ, which essentially mapps the QH states to superconducting states of the composite bosons. Most remarkably, Shahar *et al.*[14] have found that the longitudinal current–voltage characteristics $I_x(V_{xx})$ in the QH and insulating phases are related by a "reflection" symmetry, i.e. a symmetry to trading the roles of current and voltage. It was shown[14,15] that this symmetry can be interpreted as evidence for charge-flux duality relating the charged composite bosons in one phase to vortices in the other. This interpretation relies on the mapping of the observable resistivity tensor ρ_{ij} to the fictitious resistivity tensor of the composite bosons, ρ^b_{ij}:

$$\begin{aligned} \rho_{xx} &= \rho^b_{xx} \, , \\ \rho_{xy} &= \rho^b_{xy} + k\frac{h}{e^2} \, , \end{aligned} \tag{2}$$

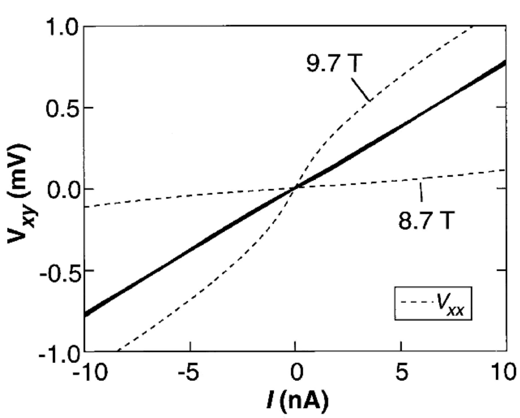

FIG. 1: Plots of the Hall voltage V_{xy} versus the current I taken at 0.1-T steps in the B range from 8.7 T to 9.7 T (the transition from $1/3$–QH state to the insulator occurs at $B_c = 9.1$). Dashed traces are of V_{xx}. $T = 21$mK. [Figure taken from Ref. 14].

a relation which has been argued to be valid beyond linear response.[15]

By inspection of the "resistivity law" [Eq. (2)] it is easy to see that the reflection symmetry

$$\rho_{xx}(\nu) = 1/\rho_{xx}(\nu_d) \tag{3}$$

(where ν and ν_d are filling factors in the QH and insulating phases, respectively, and units where $h/e^2 = 1$ are used) is unambiguously equivalent to the duality relation in the composite bosons representation

$$\rho^b_{ij}(\nu) = \sigma^b_{ji}(\nu_d) \tag{4}$$

only provided ρ^b_{ij} is diagonal, i.e. $\rho^b_{xy} = 0$. Eq. (2) then leads to the unavoidable conclusion that the observable Hall resistivity should be $\rho_{xy} = kh/e^2$ *in both the QH and insulating phases*. Thus, based on "circumstantial evidence", it has been predicted that the Hall resistivity in the insulator neighboring a particular $1/k$–QH state should remain *quantized* at the plateau value. This intriguing behavior named a "quantized Hall insulator" (QHI) was indeed confirmed experimentally by Shahar *et al.*,[14] who studied the transition from a $1/3$–QH state to the insulator in a moderately disordered GaAs sample. As indicated in Fig. 1, all traces of V_{xy} vs. I, including those taken in the insulator ($B > B_c$ where B_c is the critical field), collapse on a single, linear trace with a quantized slope: $\rho_{xy} = V_{xy}/I = 3h/e^2$. Moreover, the quantization in the Hall response has been observed in a regime where the longitudinal current–voltage relation is non–linear.

Subsequent experimental studies of ρ_{xy} in the vicinity of other QH transitions, and in different types of samples and materials, have confirmed this observation.[16,17,18] For example, Fig. 2 presents the transition from the integer $\nu = 1$ QH phase to the insulator in a two–dimensional (2D) hole system confined in a Ge/SiGe quantum well. Note that to

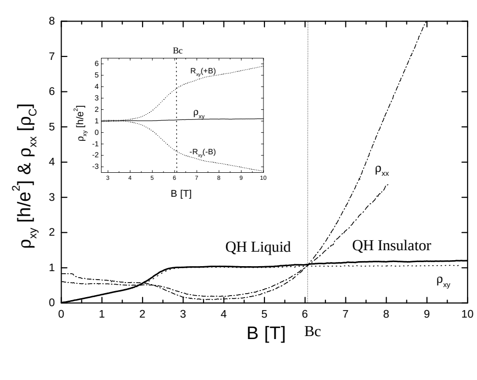

FIG. 2: The Hall and diagonal resistivities as a function of B-field. The solid line is the Hall resistivity measured at T=300 mK and a current I=200 nA, whereas the dotted line is for I=400 nA. The dash-dotted lines are ρ_{xx} at T=1.2 K (the uppermost curve) and at T=4.2 K (the lower curve). V_G=5.2 V, $\rho_c = 1.65h/e^2$ and B_C=6.06 T. The inset shows the Hall resistances for $+B$ and $-B$ in dotted lines and ρ_{xy} with a solid line. [Figure taken from Ref. 17].

eliminate the mixing with the longitudinal component of the resistivity (which is particularly large in the insulator), the Hall resistance is measured in two opposite orientations of the B; ρ_{xy} is defined as the antisymmetric component: $\rho_{xy} = 1/2[R_{xy}(+B) - R_{xy}(-B)]$ (see inset of Fig. 2).

The accumulated experimental evidence for the existence of QHI states raised a number of questions. Most prominently, when combined with earlier experimental results[11,12] it appears to support the possibility that the insulating regime in the global QH phase diagram does not consist of a single type of insulator. Rather, it includes a sequence of QHI states, characterized by different values of the odd integer k, and in between regions where ρ_{xy} is not quantized, and exhibits either the classical behavior $\rho_{xy} \sim B$ or diverges. It is not clear, however, whether these are different insulating *phases* separated by critical manifolds in the phase diagram, or different transport regimes which connect to each other by a smooth cross–over. In spite of the valuable insight gained by the bosonic Chern–Simons theory and its role in the initial discovery of the QHI, it does not seem to provide tools for a more detailed study of the transport which is required to address these issues. An alternative theoretical framework, which proved to be more practical, is reviewed in the next Section.

III. THEORETICAL MODELS

The physics of the integer QH effect is described quite well in terms of the Landau level quantization of non–interacting electrons. In particular, in very strong B such that only the first Landau level is partially occupied, and in the presence of a slowly varying disorder potential, one can use a semiclassical approximation[19] in which the electrons at the Fermi level travel along equipotential contours. Scattering among equipotential contours is possible in the vicinity of saddle points in the potential via tunneling. As a result, transport in the 2D systems is effectively carried by a network of quasi one–dimensional chiral channels (edge states), and can be modeled within the Landauer–Büttiker approach.[20] This provides a convenient framework for the study of a transition from a $\nu = 1$ QH state to the insulator, where the two phases are defined according to the behavior of the transmission through the sample ($\mathcal{T}$) in the $T \to 0$ limit: $\mathcal{T} \to 1$ in the QH phase, $\mathcal{T} \to 0$ in the insulator. This description can be generalized to any of the transitions from a $1/k$–QH state to the insulator using a mapping to composite fermions.[15] As we show below, both the "reflection symmetry" of ρ_{xx} and the behavior of ρ_{xy} can be interpreted more intuitively in this approach.

A. The Single Scatterer: a Model for the Two–Terminal Configuration

The robust quantization of ρ_{xy} was demonstrated by Jain and Kivelson[21] in a QH system assuming a single saddle–point in the potential. They consider a Hall bar connected between two terminals, as depicted in Fig. 3, where the single constriction (schematically representing the saddle–point) is characterized by a transmission probability $\mathcal{T}$. In general, $\mathcal{T}$ depends on the potential, the filling factor (where here $0 < \nu < 1$) and possibly (in the non–linear response regime) on the current bias $I = I_R - I_{L'}$. The currents in Fig. 3 are related by

$$I_{L'} = \mathcal{T} I_L + \mathcal{R} I_R \ , \tag{5}$$

where $\mathcal{R} = 1 - \mathcal{T}$ is the reflection probability.

Away from the constriction there is no scattering, and a local equilibrium is assumed so that the chemical potential and the current are related by[22]

$$\mu_i = \frac{h}{e^2} I_i \ , \quad \text{where} \quad i = R, R', L, L' \ . \tag{6}$$

The longitudinal voltage measured between the two terminals is then given by

$$V_{xx} = \mu_{L'} - \mu_L = \frac{\mathcal{R}}{\mathcal{T}} \frac{h}{e^2} I \ . \tag{7}$$

At the same time, the Hall voltage is given by

$$V_{xy} = \mu_R - \mu_{L'} = \text{sgn}(B) \frac{h}{e^2} I \ . \tag{8}$$

The factor sgn(B) accounts for the fact that a reversal of B for a fixed current I (which correspond to reversing the direction of local currents I_i and redefining the current as $I = I_{L'} - I_R$) reverses V_{xy} [i.e. $V_{xy}(B) = -V_{xy}(-B)$], while leaving V_{xx} unchanged. This implies that although the voltage measured between arbitrary two points across the Hall bar may include a longitudinal component (e.g., $\mu_R - \mu_L = V_{xx} + V_{xy}$), the antisymmetrization with respect to $B \to -B$ eliminates V_{xx}, yielding the pure Hall resistance $\rho_{xy} = V_{xy}/I = h/e^2$.

Two points should be emphasized with regards to the above result. First, ρ_{xy} is quantized regardless of any details of the scatterer: the dependence on the potential, magnetic field and bias affect the pure longitudinal response alone. In particular, the quantization survives in the insulating regime corresponding to $\mathcal{T} \to 0$, where $\rho_{xx} \to \infty$. The second interesting point is that an exchange of $\mathcal{T}$ and $\mathcal{R}$ (a procedure which corresponds to the trading of particles and holes compared to the half–filled Landau level), maps ρ_{xx} on $1/\rho_{xx}$ [see Eq. (7)]. The charge–flux duality, which has been argued to be responsible for the reflection symmetry within the bosonic Chern–Simons theory, is therefore equivalent to a particle–hole symmetry in the "fermionic" picture.[15]

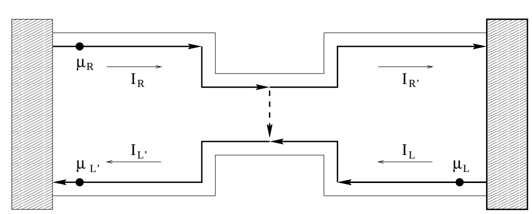

FIG. 3: A schematic illustration of a Hall bar connected to two reservoirs (shaded regions).

The above analysis can be easily generalized to the more realistic case of a macroscopic QH system, where the disorder potential includes a multitude of saddle–points with random parameters. It is still possible to define an overall transmission probability across the Hall bar connected between the terminals. This suggests, that the single–scatterer model provides a complete transport theory, which explains both the reflection symmetry and the QHI phenomenon quite directly. As a matter of fact, the latter appears to be rather trivial! However, one should keep in mind that the measurement configuration depicted in Fig. 3 is not an ideal setup for the probing of the resistivity tensor in a 2D Hall bar. The system is connected between two terminals only, each corresponding to a single pair of outgoing and incoming conducting channels. The Hall voltage as defined in such a connection (where the Hall probes are attached to the very same outgoing and incoming channels), is indeed trivially related to the current. In contrast, as we show in the next Sections, an "honest" measurement of V_{xy} which involves *four* independent contacts, is more informative. In particular, it does not necessarily yield a quantized ρ_{xy}.

B. The "Classical" Puddle–Network Model

The behavior of a most general integer QH system at $0 < \nu < 1$ is best captured by network models, e.g. of the type proposed by Chalker and Coddington (CC).[23] The sample is represented by a network of QH puddles encircled by chiral edge channels, which are connected to each other by constrictions of the type depicted in Fig. 3, each characterized by transmission and reflection amplitudes t and r. The latter are assumed to be random variables.

A study of the properties of local currents in such a network model[24] leads to the derivation of a "semi–circle" law for the conductivities in the vicinity of a transition from the $n+1$ to the n QH state

$$\sigma_{xx}^2 + \left[\sigma_{xy} - (n+1/2)\frac{e^2}{h}\right]^2 = \left[\frac{1}{2}\frac{e^2}{h}\right]^2 , \qquad (9)$$

which in the special case $n = 0$ (($\nu = 1$)–QH state to the insulator) is essentially equivalent to the statement that ρ_{xy} is quantized. This derivation is restricted to linear response and relies on the existence of a local conductivity tensor. The quantization of the Hall resistance was proved to be more robust, and in particular valid beyond linear response, in a later work[25] employing the Landauer approach to transport which allows a direct derivation of the resistivity rather than conductivity tensor. Below we sketch this derivation.

Consider a general network of QH puddles at filling factor $1/k$ (with k an odd integer) separated by constrictions (tunnel barriers), with two current leads at $-x$ and $+x$, and two voltage leads at $-y$ and $+y$ [see Fig. 4(a)]. Assuming that the typical distance between adjacent constrictions is sufficiently large to establish local equilibrium, it is possible to define for each constriction connecting puddle i to j a local Hall voltage

$$V_{xy}(ij) = \text{sgn}(B)\frac{h}{e^2}kI_{ij} , \qquad (10)$$

where I_{ij} is the current flowing from i to j. This is essentially a generalization of Eq. (8) to the fractional QH case $k \neq 1$. In addition, it is assumed that all quantum interference effects take place within the tunnel barrier length scales, beyond which dephasing effects destroy coherence between tunneling events. As a consequence, the longitudinal response of the network can be modeled by the equivalent circuit depicted in Fig. 4(b). Namely, it behaves as a classical 2D resistor network in which vertices represent the QH puddles, and the resistors mimic the constrictions between them, each characterized by a local current–voltage relation $V_{ij}(I_{ij})$. For a given total current I driven between $-x$ and $+x$, the distributions of $\{V_{ij}\}$ and $\{I_{ij}\}$ in the network is dictated by the classical Kirchoff's laws. It can be proved,[25] that the total number of equations imposed by this set of constraints is exactly sufficient to determine $\{I_{ij}\}$ uniquely.

The total transverse voltage V_y can be evaluated by choosing any path $\mathcal{C}$ which connects the $-y$ lead to the $+y$ lead (see Fig. 4). Summing up all contributions one obtains

$$V_y(B) = \sum_{(ij)^L \in \mathcal{C}} V_{ij}(I_{ij}, |B|) + \text{sgn}(B)k\frac{h}{e^2}\sum_{(ij)^H \in \mathcal{C}} I_{ij} , \qquad (11)$$

where $(ij)^L$ denote sections of the path along the resistors, and $(ij)^H$ the sections connecting two points across vertices in the graph. Here we account for the fact that V_{ij} typically depend on B, but obeys $V_{ij}(B) = V_{ij}(-B)$. Eq. (11) indicates quite transparently that the longitudinal and Hall components of the voltage drop between any two points are conveniently separated by symmetry under reversal of sgn(B). In particular, defining the Hall voltage to be the antisymmetric component $V_{xy} = 1/2[V_y(B) - V_y(-B)]$, it is easy to see that the contribution from the first term in the expression for V_y cancels out. Since by global current conservation the second term

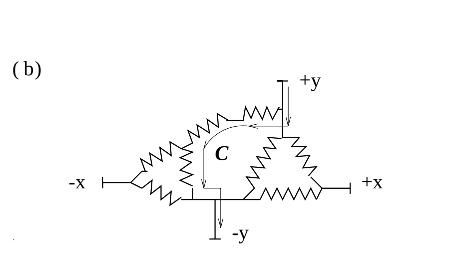

FIG. 4: (a) Typical puddle network, where dotted lines represent constrictions. (b) Corresponding equivalent circuit. Path $\mathcal{C}$ is denoted by arrows. [Figure taken from Ref. 25].

is proportional to the total current I, one obtains

$$V_{xy} = \mathrm{sgn}(B) k \frac{h}{e^2} I \ . \tag{12}$$

The Hall resistivity is therefore quantized at $\rho_{xy} = k(h/e^2)$, and *completely independent* of B and I.

The above model for the transport implies that a quantized Hall resistivity is a remarkably robust feature of any QH system in both sides of the transition from a $1/k$ QH liquid to the insulator. All details of the disorder potential, the dependence on filling factor and temperature as well as deviations from linear response are reflected by the longitudinal response alone. In particular, the quantization persists in both the QH liquid and insulating phases, in agreement with the experimental observations.[14,16,17,18] However, at this point it should be emphasized that the proof (which incorporates the use of the classical Kirchoff's laws) relies on a crucial assumption: that there is a strong dephasing mechanism, responsible for the suppression of quantum interference anywhere except the close vicinity of the tunnel barriers between adjacent puddles. This scenario is valid as long as the typical puddle–size L_p is large compared to the dephasing length L_ϕ. The condition $L_p > L_\phi$ is expected to be violated, either at sufficiently low T (where L_ϕ diverges), or due to the reduction of puddle–size sufficiently far from the transition to the insulator (or in the presence of a short–range impurity potential). The question arises, whether the requirement of complete dephasing is indeed necessary. As discussed in the next subsection, it turns out that answering this question is not straightforward.

C. The Role of Quantum Interference: Breakdown of the QHI

In the purely quantum, phase–coherent limit, electron transport in a network of the type depicted in Fig. 4(a) is no longer dictated by Kirchoff's laws. Local currents in the network are given by summing current *amplitudes* rather than current *probabilities*. Such a coherent summation introduces interference terms which are likely to have dramatic influence on the transport. In particular, interference effects are known to be responsible for the emergence of localization. Obviously, the transport problem becomes far less tractable in this limit. Indeed, attempts to establish a theoretical description of the transport coefficients, and the behavior of the Hall resistance in particular, were subject to controversies over the last few years.

The quantum limit was first addressed by Ruzin and Feng,[24] who argued that the semi–circle law Eq. (9) (and hence, in particular, the quantization of ρ_{xy} in the insulator neighboring a QH state) is still valid in this limit. Their derivation employs a description of the QH system in terms of the chess-board-like (CC) version of the network model.[23] Defining the local currents in adjacent patches (representing the QH states n and $n+1$) to be J_1 and J_2, respectively, their proof of Eq. (9) proceeds in two stages. First, they prove the equivalence of the semi–circle law to the statement that J_1 and J_2 are orthogonal to each other. Then, they demonstrate the orthogonality of the currents. The latter property is shown to be valid in the quantum limit as well, using a numerical computation. However, the proof of the first part relies on the assumption of local equilibrium. Hence, the statement of equivalence to the semi–circle law is not clearly applicable in the quantum coherent limit.

In a later study, Pryadko and Auerbach[26] have evaluated the Hall resistance of a finite quantum coherent CC–network *directly*. They have computed numerically the complex amplitudes associated with local currents in a network, where the transmission and phase characterizing coherent scattering at the nodes are random variables. Assuming the network to be measured by four separate leads labeled 1 through 4 (each lead connected to a single pair of incoming and outgoing chiral edge channels), the Hall voltage for a given total current driven between leads 1 and 2 is determined (exactly as in the previous subsection) as the antisymmetric component of $V_y(B) = (\mu_3 - \mu_4)$. The local chemical potentials μ_i are related to the corresponding local currents through Eq. (6), since at the leads a local equilibrium is established. The result of this calculation are radically different from the classical case [Eq. (12)]: the Hall resistance in the insulator *diverges* exponentially with the system size L (see Fig. 5). Moreover, ρ_{xx} and ρ_{xy} scale with each other: $\rho_{xy} \sim \rho_{xx}^{\gamma}$, where $\gamma \approx 1/3$.

Another numerical study of the coherent transport in a QH system was performed by Sheng and Weng[27] within a tight–binding model of (non–interacting) electrons in a strong magnetic field at filling factor $\nu < 1$. Using different numerical procedures, they have computed the conductivities σ_{xx} and σ_{xy} and determined the resistivities by tensor inversion. Despite significant quantitative differences compared to Ref. 26, the insulating regime exhibits the same behavior: an asymptotic scaling relation correlations between ρ_{xx} and ρ_{xy} ($\rho_{xy} \sim \rho_{xx}^{\gamma}$), both of which diverge exponentially with the system size (the exponent γ is, however, different ($\gamma \sim 1$), possibly indicating a model–dependence of its particular value). This behavior is demonstrated in Fig. 6 for $\rho_{xx} > 10h/e^2$. It should be noted, however, that in a wide range of parameters beyond the QH–to–Insulator transition (which occurs at $\rho_{xx}^c \approx h/e^2$), a QHI behavior is indicated. The Hall resistance deviates from

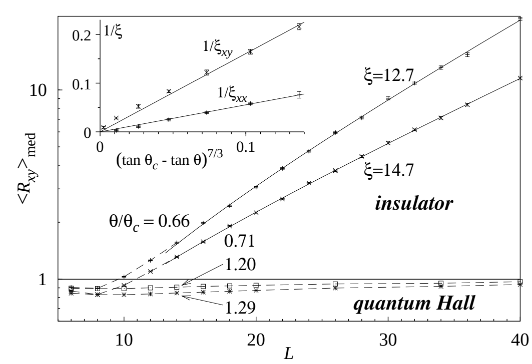

FIG. 5: Finite size scaling of the Hall resistance in the QH and insulator phases. ξ denotes the localization length, where ξ_{xx} and ξ_{xy} correspond to ρ_{xx} and ρ_{xy}, respectively. [Figure taken from Ref. 26].

the quantized value h/e^2 and starts diverging only deep in the insulating phase, where the system size L is sufficiently large compared to the localization length ξ (so that ρ_{xx} is at least an order of magnitude larger than ρ^c_{xx}).

The finite size behavior obtained in the numerical studies described above effectively simulates a physical macroscopic system (where $L \to \infty$) at finite temperature T: in that case the finite size cutoff L is replaced by a characteristic T–dependent dephasing length L_ϕ. The numerical data indicate that the QHI behavior breaks down at sufficiently low T such that $L_\phi \gg \xi$, thus raising the intriguing possibility that the breakdown originates from quantum interference, manifested in terms of localization effects.

To gain more insight into the underlying localization mechanism, Zülicke and Shimshoni[28] have studied a model of the QH system which enables the analytical derivation of scaling expressions for the transport coefficients ρ_{xx} and ρ_{xy}. They consider electron transport on a random network that is constructed as a hierarchical lattice (see Fig. 7). The random variables are the transmission and reflection amplitudes at the vertices of the elementary cells and the phases accumulated in closed loops.

Scaling equations relating the resistivity tensor components to the number of stages in the hierarchy (n) are now derived using a real–space renormalization group (RG) procedure, which provides a 2D generalization of the familiar scattering approach to 1D localization.[29] Note that a similar applications of hierarchical structures have been used to study the critical behavior near a QH transition.[30] Here, the focus is on the asymptotic behavior of transport coefficients deep in the QH and insulating phases rather than at the critical point. The procedure is as follows: at each RG step, the effective transmission and reflection probability of a 4–vertex cell in the nth level of the hierarchy ($T^{(n)}$, $R^{(n)}$) are expressed in terms of $T_i^{(n-1)}$, $R_i^{(n-1)}$ ($i = 1, 2, 3, 4$) and the phases in the previous level. Then, similarly to Anderson *et al.*,[29] one identifies quantities that are statistically well–behaved (i.e., their distribution obeys the central limit theorem), so that their statistical averages over the random variables dictate the *typical* values of various meaningful physical quantities. In the present case, the distributions of $\ln T^{(n)}$ and $\ln R^{(n)}$ are asymptotically normal (i.e. for large n), and their average values determine the scaling expressions for the total current, the Hall and longitudinal voltages in a system of linear size $L = 2^n$.

Before stating the results of this calculation, it is useful to point out that the study of 1D localization[29] already provides much insight on the expected behavior of a generic QH network. Consider an extreme case of a very long and narrow Hall bar, where N scatterers are connected serially in a chain (Fig. 8). According to Ref. 29 the typical transmission probability of the entire chain, given by $T = T_0^N$ where $T_0 = \exp\{\langle \ln |t_j|^2 \rangle_{t_j}\}$, vanishes exponentially with the system size; consequently, the typical longitudinal resistance $\rho_{xx} = R/T(h/e^2)$ diverges exponentially:

$$\rho_{xx} \sim \frac{1}{T_0^N}\frac{h}{e^2} \ . \tag{13}$$

Consider now the opposite extreme case of a very short and wide Hall bar, where the very same N scatterers are connected *in parallel*. This scenario is also depicted by Fig. 8 provided the total current flows *vertically downwards*, the coefficients t_j represent the *reflection* amplitudes of the scatterers, and the roles of T and R are exchanged. It is immediately implied that the system exhibits exponential *delocalization*, i.e. the

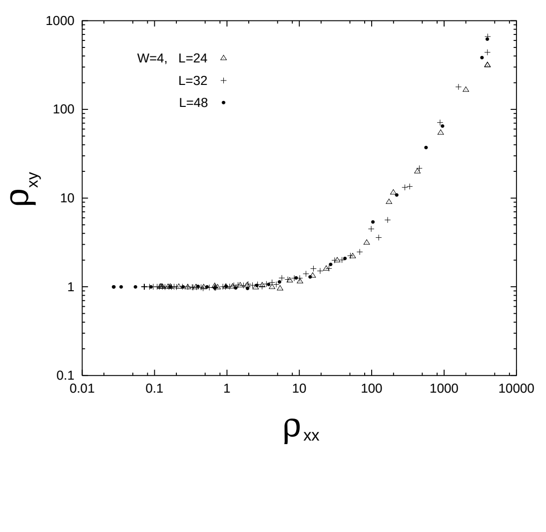

FIG. 6: ρ_{xy} vs. ρ_{xx} for fixed disorder strength (denoted by W). [Figure taken from Ref. 27].

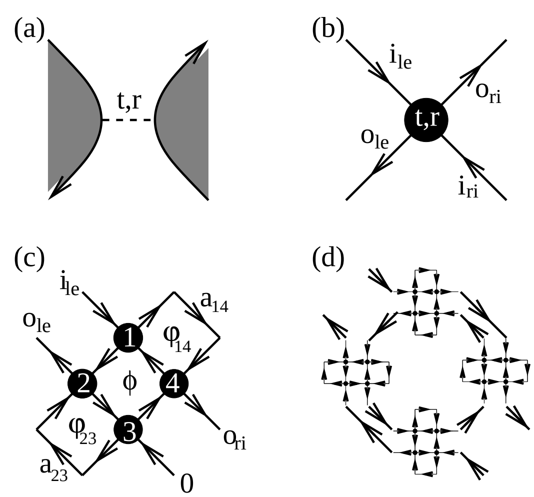

FIG. 7: The hierarchical network model. (a) A saddle point between adjacent electron puddles. Motion within a puddle is directed along chiral edge channels. (b) Saddle point represented by a scattering matrix (vertex) relating outgoing electron amplitudes (o_{le}, o_{ri}) to incoming ones (i_{le}, i_{ri}). (c) Elementary cell of the hierarchical network. Quantum phases acquired by electrons when moving around its three small closed loops are indicated. (d) Lattice at the second level of the hierarchy.

longitudinal resistance *vanishes* as

$$\rho_{xx} \sim T_0^N \frac{h}{e^2} \ . \tag{14}$$

In a general, truly 2D QH network, serial and parallel connections of scatterers are completely intertwined and equally abundant. The longitudinal transport through a macroscopic system is therefore expected to exhibit either a perfect conducting behavior or an insulating behavior depending on the value of T_0. In particular, a localization–delocalization transition is expected close to the symmetric point $T_0 = 1/2$. We hence deduce that a generic QH network in the quantum coherent limit is characterized by a "duality symmetry" between the QH and insulator phases, of the same origin as the single constriction: it is associated with a symmetry to trading of transmission and reflection coefficients combined with the trading of serial and parallel connections. It is important to emphasize that this symmetry property is not related in any way to the behavior of the Hall resistance in the two phases.

The RG analysis of Ref. 28 indeed accounts for the 2D nature by introducing a serial and parallel connection at each step. The resulting asymptotic expression for the typical longitudinal resistivity in the limit of large $L = 2^n$ is

$$\rho_{xx} \approx \left(\frac{\tilde{R}_0}{\tilde{T}_0}\right)^L \frac{h}{e^2} \ , \tag{15}$$

where $\tilde{T}_0$ [$\tilde{R}_0$] are related[28] to the log–average of the transmission [reflection] coefficients in the initial distribution (for $n = 0$) and its standard deviation. This indicates a transition from a conducting (QH) state ($\rho_{xx} \to 0$) to an insulator ($\rho_{xx} \to \infty$) at $\tilde{R}_0 = \tilde{T}_0$, thus confirming the expectation based on the naiive extrapolation of Ref. 29. The behavior of the typical Hall resistivity is much harder to predict. The calculated ρ_{xy} recovers the quantized value in the QH phase. However, in the insulator it diverges exponentially:

$$\rho_{xy} = \frac{h}{e^2}\left[1 - \left(\tilde{R}_0\right)^{2^n}\right] / \left(\tilde{T}_0\right)^{2^{n-1}} \approx \frac{h}{e^2}\left(\tilde{T}_0\right)^{-L/2} \ . \tag{16}$$

Note that similarly to the numerical data,[26,27] in the strongly insulating limit ($\tilde{T}_0 \ll \tilde{R}_0$) ρ_{xx} and ρ_{xy} obey a scaling relation $\rho_{xy} \sim \rho_{xx}^{\gamma}$ where here $\gamma \approx 1/2$. The corresponding localization lengths are given by

$$\xi_{xy} = 2\xi_{xx} \ , \quad \text{where} \quad \xi_{xx} \approx -\frac{1}{\ln \tilde{T}_0} \ . \tag{17}$$

The scaling behavior obtained in Ref. 28 supports the central conclusion of the numerical studies[26,27] – namely, it indicates a destruction of the QHI by quantum coherence effects.

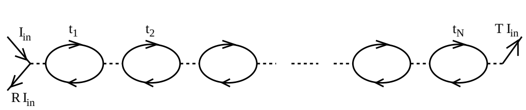

FIG. 8: A serial chain of scatterers in a Hall bar

Moreover, intermediate stages of the analysis[28] already reflect a radical difference in behavior of the Hall resistance between the QH and insulating phases. This is in sharp contrast with the classical transport regime. On the other hand, the analysis points at certain difficulties which call for further investigation. In particular, it is observed that even the logarithmic distribution of ρ_{xy} is quite wide and the RG equations flow to the asymptotic form quite slowly. In addition, the particular structure of the hierarchical network (Fig. 7), which was chosen to facilitate the analytical calculation, effectively favors serial connections over parallel ones in the configuration assumed for the elementary cell. As a result the duality symmetry is violated: in spite of the "symmetric appearance" of ρ_{xx} [Eq. (15)], the critical point for a transition from the QH phase to the insulator is underestimated. To see that, note that the universal value $\rho_{xx} = h/e^2$ is established at $\tilde{R}_0 = \tilde{T}_0$, however $\tilde{T}_0$ is larger than T_0, the typical transmission probability in the initial distribution of scatterers, implying that the transition occurs at $T_0 < 1/2$. Such a bias is not characteristic of a generic QH system.

A subsequent work by Cain and Römer[31] addresses these problems. First, they introduce a hierarchical network where the elementary cell includes a fifth scatterer. The same structure was used earlier by Galstyan and Raikh[30] and by Cain *et al.*[32] in their RG studies of the critical exponent, level statistics and long–range correlations of the disorder potential in the vicinity of the transition. Apparently, this structure restores the symmetry and yields numerical values for the critical exponent which are very close to other numerical procedures. In addition, a numerical application of the real–space RG iterations yields the asymptotics of the entire distributions $P(R_{xx})$, $P(R_{xy})$ of both the longitudinal and Hall resistance components. Identifying ρ_{xy} with either the most probable value of $P(R_{xy})$ or the typical value, they observe a QHI behavior up to $\rho_{xx} \sim 25h/e^2$. Deeper in the insulator, ρ_{xy} diverges and obeys the scaling relation $\rho_{xy} \sim \rho_{xx}^{\gamma}$ with $\gamma = 0.26$. Particular details of the crossover between the two regimes are very sensitive to the averaging procedure. Note that the numerical value of γ is close to the exponent found in Ref. 26, possibly reflecting the fact that the network models studied in the two cases are essentially equivalent.

The theoretical works reviewed above span a variety of different models and different approaches to the evaluation of resistivity tensors in a macroscopic QH system. Yet, they converge to a remarkably unique conclusion: that the strictly $T = 0$ insulating phase in the QH regime is *not* a QHI. Rather, quantum interference effects induce localization, and divergence of both components of the resistivity tensor. The scaling relation $\rho_{xy} \sim \rho_{xx}^{\gamma}$ implies the existence of a single localization length ξ, to which ξ_{xx} and ξ_{xy} are related by a numerical factor – apparently dependent on the geometry of the disorder potential. In finite size systems, the QHI behavior is typically observed in a wide but limited range of parameters beyond the critical point, and essentially does not survive in the thermodynamic limit. The unavoidable conclusion is that in a realistic system at finite T, it does not characterize the insulating phase but rather a regime of classical transport, where $L_\phi < \xi$ thus suppressing quantum interference.

IV. RECENT EXPERIMENTAL RESULTS AND OPEN QUESTIONS

The theoretical studies predicting an asymptotic breakdown of the QHI also imply that there are severe restrictions on the experimental observation of this breakdown. A significant deviation of ρ_{xy} from the quantized plateau value is expected in the strong localization regime where $L_\phi \gg \xi_{xy}$. Since typically the divergence of ρ_{xy} is relatively moderate ($\xi_{xy} > \xi_{xx}$), it appears that this quantum regime is approached only when the longitudinal resistivity ρ_{xx} is more than an order of magnitude larger than its critical value of $\sim h/e^2$. The appropriate conditions can be achieved at very low T and strong magnetic fields, far enough from the critical value B_c. In addition, better conditions for localization are established in highly disordered samples. This regime of parameters is accessible; however, the main problem is that it is hard to obtain reliable experimental data deep in the insulating phase. In particular, due to the wide distribution of the Hall resistance,[31] its measured value is expected to be very sensitive to the measurement procedure and the external circuitry.

A recent experimental study[33] has focused on magnetotransport measurements in GaAs/GaAlAs heterostructures that are deliberately contaminated by Be acceptors in a δ–doping layer. These introduce a short range disorder, hence expected to reduce the localization length in the insulating state. Preliminary data obtained at low T (down to T = 60mK) indicate a considerable deviation of ρ_{xy} from the quantized value h/e^2 when ρ_{xx} increases beyond 100KΩ. Deeper in the insulating regime, where $\rho_{xx} > 10h/e^2$, a divergence of ρ_{xy} (up to $\sim 20h/e^2$) is observed, and the scaling relation $\rho_{xy} \sim \rho_{xx}^{\gamma}$ is approximately obeyed. Due to the subtleties of the measurement setup the error bars on ρ_{xy} and hence on the numerical value of γ are relatively large (γ is estimated in the range 0.5 to 0.75), however the divergence of ρ_{xy} far beyond the quantized value appears to be conclusive.

Yet another recent experimental result[34] can possibly be interpreted as a challenge to the theoretical predictions in the quantum coherent limit. Magnetotransport measurements near the transition from a $\nu = 1$ QH state to the insulator were performed on small samples of low–mobility InGaAs/InAlAs (of a few microns in each dimension) at low T. The trace of ρ_{xx} as a function of the magnetic field B was found to display strong reproducible fluctuations, whose amplitude increases when the sample size is reduced. In contrast, ρ_{xy} in the same range of parameters is quantized to a very good approximation. Since the fluctuations appear to be highly reminiscent of universal conductance fluctuations (UCF) that are known to reflect the quantum coherence nature of transport in mesoscopic samples, this observation suggests that quantum interference effects are present yet the QHI behavior is not destroyed. Rather, the fluctuations in the component σ_{xx}, σ_{xy} of the conductivity tensor are correlated in such a way that their fluctuations conspire to cancel each other in the expression for ρ_{xy}. A similar correlation was also identified near the transition between QH plateaus.[35]

There are two possible interpretations for these experimental results. The first option assumes that (even at the low T tested in these measurements) L_ϕ is sufficiently small to impose the conditions of classical transport in the system. In that case, ρ_{xy} is obviously quantized, and ρ_{xx} is dictated by the classically connected network of resistors associated with junctions between QH electron puddles. The observed fluctuations in ρ_{xx} are then a consequence of the statistical distribution of $\log[R_{xx}]$ in a small system where the number of "resistors" ($N \approx L/l_{el}$, where L is the linear size of the system and l_{el} the elastic mean free path) is far from the thermodynamic limit. In the present case, $l_{el} \sim 0.1\mu$m implying that in the smallest measured samples $\sqrt{N} \approx 20$. Indeed, the average size of the fluctuations is consistent with the expected standard deviation. However, the spiky nature of the ρ_{xx} trace as a function of B is not consistent with the B–dependence expected from variations in the width of tunnel–barriers between QH puddle. The rapid fluctuations can be alternatively attributed to charging effects involved in the hopping processes between QH puddles, a mechanism suggested following earlier experimental observations of mesoscopic fluctuations.[36]

The second possibility is that the fluctuations indeed reflect the effect of a magnetic field on the interference pattern between different electron paths in a quantum coherent transport regime. This mechanism could result in reproducible fluctuations of a similar character as UCF, although their amplitude should not be necessarily universal (more likely, it is dictated by the typical value of the resistivity). According to the theories described in subsection 3.3, such interpretation is apparently in conflict with the QHI behavior.

It should be noted, however, that the present theoretical understanding is capable of definite predictions in either of two cases. In the extreme "classical transport" limit, where L_ϕ is smaller than a puddle size, interference is entirely suppressed and ρ_{xy} is quantized.[25] In the extreme "quantum transport" limit, where L_ϕ is larger than the localization length ξ, quantum interference destroys the quantization in the insulator and leads to a divergence of ρ_{xy} in the thermodynamic limit.[26,27,28,31] The puddle size is of order l_{el}, and since close to the transition ξ is typically much larger than l_{el} there is an intermediate range where the theory is practically not decisive about the behavior of ρ_{xy} (especially in view of its wide distribution). The experimental data,[34] which mostly accumulate within this intermediate range (where ρ_{xx} increases by at most one order of magnitude), possibly provide evidence for the stability of the QHI behavior in the entire region where $\xi > L_\phi$.

A better theoretical understanding of the transport behavior in the above mentioned intermediate regime would require a detailed study of the mechanism leading to destruction of phase coherence, and correspondingly a concrete evaluation of the dephasing length L_ϕ. In section 3 it was essentially defined as the typical length scale over which the interference between different electron trajectories is suppressed. Dephasing is provided by coupling of the system to other degrees of freedom and by electron–electron interactions, hence L_ϕ generally depends on many details such as the strength of various coupling constants. However, it is possible that close enough to the transition and at sufficiently low T, some of the dephasing processes become irrelevant and the dynamics is dictated

by universal properties, signifying the quantum critical nature of the transition point.[9,37] L_ϕ should then be identified with the correlation time $\xi_T \sim \xi^z$ (with z the dynamical exponent), and the crossover line $\xi \sim L_\phi$ (which apparently marks the breakdown of the QHI) is to be identified with the borderline of the quantum critical regime.[38] Unfortunately, a convenient field–theoretical model which enables a direct analysis of the transport properties at finite T within this framework is presently lacking.

On the experimental front, further investigation is required to distinguish conclusively between the different possible interpretations of the fluctuations in mesoscopic samples. For example, the study of the effect of a screening gate could test the significance of Coulomb blockade; a Fourier analysis of the B–dependence of the fluctuations in ρ_{xx}, and the study of correlations between the typical periodicity in magnetic flux and the scale l_{el}, may shed light on the role of quantum interference.

V. CONCLUSIONS

As demonstrated in a multitude of experiments reviewed in this article, the insulating regime neighboring a primary QH liquid state $\nu = 1/k$ (with k an odd integer) exhibits a QHI behavior: a quantized Hall resistance accompanied by an insulating–like character of the longitudinal transport. The phenomenon has been studied theoretically using various models, and in particular network models for the transport in the QH regime. The results of these studies indicate that the QHI does not characterize the full–fledged insulating phase, which is established sufficiently far from the transition. Nor is there any evidence for an additional phase transition (from a QHI to a "true" insulator) within the insulating regime. Rather, it is predicted that in the $T = 0$ insulating phase, where transport is dominated by quantum interference, the Hall resistance should diverge. The QHI behavior is characteristic of a "classical" transport regime, where quantum coherence on a scale larger then l_{el} (a typical size of a QH puddle) is suppressed by a dephasing mechanism. Transport in this regime is therefore best modeled as a resistor network, where the individual resistors exhibit quantum features, yet the connections between them obey the *classical* Kirchoff's laws. Preliminary experimental data support the breakdown of the QHI deep in the insulator, where localization on length scales larger than ξ takes place. There is, however, recent evidence for a robust quantization of the Hall resistance in the entire regime where ξ is of order or larger than the dephasing length L_ϕ. It is suggestive that this regime can be identified as a quantum critical regime, yet a full understanding of the transport mechanism requires further investigation.

Finally, a comment is in order regarding the initial observation of a connection between the QHI behavior and duality symmetry.[14,15] Later studies certainly support the suggestion that in the quantum critical regime both features coincide. However, in view of the network models of the transport in the QH and neighboring insulator phase, it appears that the two phenomena are actually distinct. Duality symmetry is associated with the symmetry (on average) to curvature inversion of the saddle–points in the disorder potential (and thus to trading transmission and reflection coefficients), and does not depend on the transport being quantum coherent or classical. In contrast, the lack of quantum coherence and the localization effects derived from it is crucial for the QHI phenomenon. Duality symmetry in the longitudinal transport is possible even in a regime where the Hall resistance in the insulator is not quantized. In principle, in samples where particle–hole symmetry around the critical filling factor is strongly violated, duality symmetry may not be obeyed, yet the QHI behavior is established as long as quantum interference in the global transport is suppressed. This leads to an interesting conjecture about the set of arguments previously leading to the conclusion that the two features are interwined within the composite bosons picture (see Section 2) – e.g., the application of the "resistivity law" [Eq. (2)]. It is possible that they actually rely on an implicit assumption that the transport properties are classical, Boltzmann–like in nature.

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